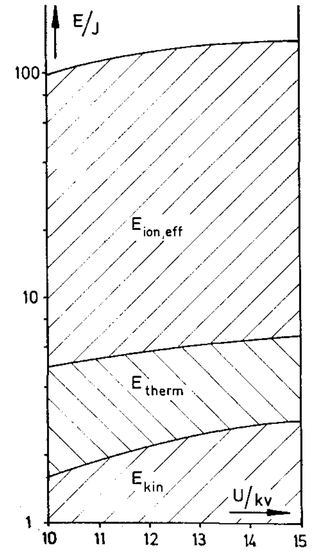

Fig. 5 Energy balance in function of capacitor voltage.

number of 2×10^{18} and 1×10^{18} for the 1st and 2nd plasmoid, respectively, is obtained.

In Fig. 5, the resulting energy balance is presented as a function of U. The effective ionization energy $E_{\text{ion,eff}}$ was calculated using Eq. (2).

Discussion

This investigation did not serve so much as to study the acceleration of a plasma, but rather to get information on a continuously working plasma source with a well-defined initial plasma velocity v. It can be seen from Fig. 2 that v varies slowly with U. As known from earlier investigations, the oscillating frequency f has a strong influence on v. This will be investigated by further measurements. Fig. 3 shows that an increase of E_{tot} up to a factor larger than two yields only an increase in E_{pl} of about 40%, because α_1 decreases with increasing U. The main portion of E_{pl} (Fig. 5) is needed for the ionization energy $E_{\text{ion,eff}}$ (more than 90%), whereas E_{kin} and E_{therm} attain only small values.

As seen from the energy balance (Fig. 5), $E_{\text{ion,eff}}$ per particle is about ten times the theoretical value which is 4×10^{-18} J/particle. Because of technical and physical reasons, the realizable range of particle flux $\dot{N}$ for continuously working accelerators is about $10^{20} - 10^{21}$ particles/sec. Thus, the necessary power requirement P_{pl} for such a source can be estimated approximately

$$P_{\text{pl}} \approx \dot{N}\ (\text{particles/s}) \cdot E_{\text{ion,eff}}(J/\text{particles})$$

Therefore, P_{pl} is 5 kw $\lesssim P_{\text{pl}} \lesssim$ 50 kw. The value of the effective efficiency of energy transfer η_{eff} (Eq. 7) was about 60–90% for the pulsed accelerator, so that assuming similar values for the continuous accelerator the power requirement must be divided by the corresponding η_{eff}.

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Laminated Orthotropic Plates under Transverse Loading

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Introduction

THE use of high performance composites for lightly loaded structures leads to a small number of plies, thus producing asymmetrical laminates. These plates exhibit coupling between bending and stretching as was shown by Reissner and Stavsky[1] for two layer plates. Laminates which are symmetric about the reference surface do not produce coupling and behave as homogeneous anisotropic plates.

Stavsky[2] has established a linear theory for multilayer aeolotropic plates, based on the Kirchhoff hypothesis, and shown the reduction to a single eighth order equation for a generalized stress function. He gave solutions for cylindrical bending and uniform distribution of stress resultants and couples. Whitney and Leissa[3] have given a formulation for heterogeneous anisotropic plates and the reduction to three equations that are expressed in terms of displacements. They presented solutions for sinusoidal transverse loading, vibration, and buckling of cross-ply and angle-ply laminates. These plates have an even number of layers of the same thickness and elastic properties. The layers are alternately oriented at 0° and 90° to the plate axes for the cross-ply laminate and $\pm\theta$ for the angle-ply. Whitney[4] expanded each displacement in a double Fourier series and gave solutions for cross-ply and angle-ply plates under transverse load.

Whitney and Leissa[5] used a formulation in terms of a stress function and transverse displacement given by Dong, Pister and Taylor[6] and expanded both dependent variables in double Fourier series. They presented solutions for cross-ply and angle-ply plates under uniform transverse load, vibration and buckling. The Ritz technique was used by Ashton and Waddoups[7] to minimize the energy expression for symmetric anisotropic plates. Beam vibration modes were used for the assumed deflection functions and they studied displacement under lateral load, vibration and buckling.

This study uses the generalized stress function equation[2] and develops a Navier type of solution for simply supported, arbitrarily laminated, orthotropic plates under transverse loadings. The plate may have any number of orthotropic layers of arbitrary thicknesses and elastic properties with axes of elastic symmetry coincident with the plate axes. Deflection, stress resultants and couples are given explicitly

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and the solution reduces to that given by Lekhnitskii[8] for homogeneous orthotropic plates.

Analysis

Coupling between in-plane stretching and transverse bending is revealed in the plate constitutive equations.[2]

$$\begin{bmatrix} N \\ \hline M \end{bmatrix} = \begin{bmatrix} A & \vert & B \\ \hline B & \vert & D \end{bmatrix} \begin{bmatrix} \epsilon^0 \\ \hline \kappa \end{bmatrix} \tag{1}$$

where

$$(A_{ij},B_{ij},D_{ij}) = \int_{-h/2}^{h/2} C_{ij}(1,Z,Z^2)dZ(i,j = 1,2,6)$$

and the C_{ij}'s are elastic coefficients. The in-plane, coupling, and bending stiffness matrices, $[A]$, $[B]$ and $[D]$ respectively, are symmetric since C_{ij} is symmetric and they are of orthotropic form, that is

$$[A] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \text{etc.}$$

since the axes of elastic symmetry of each layer are coincident with the plate axes. Stress resultants (N_{ij}), stress couples (M_{ij}), mid-surface strains (ϵ_{ij}^0), and bending curvatures (κ_{ij}) are defined as is usual in plate theory based on the Kirchhoff hypothesis. These equations show that coupling vanishes ($B_{ij} = 0$) if C_{ij} is an even function of Z.

The equilibrium and compatibility equations may be expressed in terms of a stress function F and transverse displacement w as follows. Partial inversion of the plate constitutive Eq. (1) gives

$$\begin{bmatrix} \epsilon^0 \\ \hline M \end{bmatrix} = \begin{bmatrix} A^* & \vert & B^* \\ \hline C^* & \vert & D^* \end{bmatrix} \begin{bmatrix} N \\ \hline \kappa \end{bmatrix} \tag{2}$$

where

$$[A^*] = [A^{-1}],\ [B^*] = -[A^{-1}][B]$$

$$[C^*] = [B][A^{-1}] = -[B^*]^T,\ [D^*] = [D] - [B][A^{-1}][B]$$

Note that $[A^*]$ and $[D^*]$ are symmetric but $[B^*]$ and $[C^*]$ are not in general and they are all of orthotropic form. An Airy stress function F is introduced such that

$$N_{xx} = \partial^2F/\partial y^2 \quad N_{yy} = \partial^2F/\partial x^2 \quad N_{xy} = -\partial^2F/\partial x\partial y \tag{3}$$

which satisfies two of the equilibrium equations. Transverse shear resultants are eliminated from the other three equilibrium equations to produce a single equilibrium equation

$$L_1w + L_3F = p \tag{4}$$

where

$$L_1(\ ) = D_{xx}^*\partial^4(\ )/\partial x^4 + 2(D_{xy}^* + 2D_{ss}^*)\partial^4(\ )/\partial x^2\partial y^2 + D_{yy}^*\partial^4(\ )/\partial y^4$$

$$L_3(\ ) = B_{yx}^*\partial^4(\ )/\partial x^4 + (B_{xx}^* + B_{yy}^* - 2B_{ss}^*) \times \partial^4(\ )/\partial x^2\partial y^2 + B_{xy}^*\partial^4(\ )/\partial y^4$$

The compatibility equation, in terms of strains, is the same as that of homogeneous plate theory. Substituting from Eqs. (2) and (3) into it gives

$$L_2F - L_3w = 0 \tag{5}$$

where

$$L_2(\ ) = A_{yy}^*\partial^4(\ )/\partial x^4 + (2A_{xy}^* + A_{ss}^*)\partial^4(\ )/\partial x^2\partial y^2 + A_{xx}^*\partial^4(\ )/\partial y^4$$

Equations (4) and (5) show that coupling is governed by the L_3 operator which depends upon the elements of $[B^*]$.

Equations (4) and (5) may be transformed into a single eighth-order equation for a generalized stress function Φ by setting

$$F = L_3\Phi \qquad w = L_2\Phi \tag{6}$$

which identically satisfies Eq. (5) since the operators are permutable. Equation (4) then becomes

$$(L_1L_2 + L_3^2)\Phi = p \tag{7}$$

It may be normalized and written explicitly as

$$(\beta/\alpha + \omega^2)\partial^8\Phi/\partial x^8 + (\delta/\alpha + \gamma\beta + 2\chi\omega)\partial^8\Phi/\partial x^6\partial y^2 + (\alpha^{-1}\beta^{-1} + \alpha\beta + \gamma\delta + 2\omega\psi + \chi^2)\partial^8\Phi/\partial x^4\partial y^4 + (\gamma/\beta + \alpha\delta + 2\psi\chi)\partial^8\Phi/\partial x^2\partial y^6 + (\alpha/\beta + \psi^2)\partial^8\Phi/\partial y^8 = \tilde{p} \tag{8}$$

where

$$\alpha = (A_{xx}^*/A_{yy}^*)^{1/2},\ \beta(D_{xx}^*/D_{yy}^*)^{1/2}$$

$$\gamma = (2A_{xy}^* + A_{ss}^*)/(A_{xx}^*A_{yy}^*)^{1/2}$$

$$\delta = 2(D_{xy}^* + 2D_{ss}^*)/(D_{xx}^*D_{yy}^*)^{1/2}$$

$$\chi = (B_{xx}^* + B_{yy}^* - 2B_{ss}^*)/(A_{xx}^*A_{yy}^*D_{xx}^*D_{yy}^*)^{1/4}$$

$$\psi = B_{xy}^*/(A_{xx}^*A_{yy}^*D_{xx}^*D_{yy}^*)^{1/4}$$

$$\omega = B_{yx}^*/(A_{xx}^*A_{yy}^*D_{xx}^*D_{yy}^*)^{1/4}$$

$$\tilde{p} = p/(A_{xx}^*A_{yy}^*D_{xx}^*D_{yy}^*)^{1/2}$$

The type of simple support considered is such that normal displacements are permitted but tangential displacements are not. Thus, the boundary conditions are

$$\text{Along } x = 0 \text{ and } x = a\text{: } w = v = N_{xx} = M_{xx} = 0 \tag{9}$$

$$\text{Along } y = 0 \text{ and } y = b\text{: } w = u = N_{yy} = M_{yy} = 0 \tag{10}$$

A solution is obtained by expanding the transverse load $\tilde{p}$ and generalized stress function Φ into double Fourier series

$$\tilde{p} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tilde{p}_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{11}$$

$$\Phi = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Phi_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{12}$$

The boundary conditions (9) and (10) are identically satisfied by Eq. (12). Substituting Eqs. (11) and (12) into (8) and equating coefficients provides an expression for the solution coefficients

$$\Phi_{mn}[(m\pi/a)^8(\beta/\alpha + \omega^2) + (m\pi/a)^6(n\pi/b)^2(\delta/\alpha + \gamma\beta + 2\chi\omega) + (m\pi/a)^4(n\pi/b)^4(\alpha^{-1}\beta^{-1} + \alpha\beta + \gamma\delta + 2\omega\psi + \chi^2) + (m\pi/a)^2(n\pi/b)^6(\gamma/\beta + \alpha\delta + 2\psi\chi) + (n\pi/b)^8(\alpha/\beta + \psi^2)] = \tilde{p}_{mn} \tag{13}$$

The transverse displacement is obtained from Eq. (6) and it is given by

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{yy}^*\left(\frac{m\pi}{a}\right)^4 + (2A_{xy}^* + A_{ss}^*)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + A_{xx}^*\left(\frac{n\pi}{b}\right)^4\right]\Phi_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{14}$$

This expression reduces to the equation given by Lekhnitskii[8] for homogeneous orthotropic plates and it further reduces to that given by Navier for homogeneous isotropic plates. Stress resultants and couples are obtained from Eqs. (1, 2 and 6) along with the curvature displacement equations and they are

$$N_{xx} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{n\pi}{b}\right)^2\left[B_{yx}^*\left(\frac{m\pi}{a}\right)^4 + (B_{xx}^* + B_{yy}^* - 2B_{ss}^*)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + B_{xy}^*\left(\frac{n\pi}{b}\right)^4\right]\Phi_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$N_{yy} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m\pi}{a}\right)^2\left[B_{yx}^*\left(\frac{m\pi}{a}\right)^4 + (B_{xx}^* + B_{yy}^* - 2B_{ss}^*)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + B_{xy}^*\left(\frac{n\pi}{b}\right)^4\right]\Phi_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$N_{xy} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m\pi}{a}\frac{n\pi}{b}\left[B_{yx}^*\left(\frac{m\pi}{a}\right)^4 + (B_{xx}^* + B_{yy}^* - 2B_{ss}^*)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + B_{xy}^*\left(\frac{n\pi}{b}\right)^4\right]\Phi_{mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$M_{xx} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{(B_{yx}^{*2} + A_{yy}^*D_{xx}^*)\left(\frac{m\pi}{a}\right)^6 + [B_{yx}^*(B_{yy}^* + 2B_{xx}^* - 2B_{ss}^*) + D_{xx}^*(2A_{xy}^* + A_{ss}^*) + D_{xy}^*A_{yy}^*]\left(\frac{m\pi}{a}\right)^4\left(\frac{n\pi}{b}\right)^2 + [B_{xx}^*(B_{xx}^* + B_{yy}^* - 2B_{ss}^*) + B_{yx}^*B_{xy}^* + A_{xx}^*D_{xx}^* + D_{xy}^*(2A_{xy}^* + A_{ss}^*)]\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^4 + (B_{xx}^*B_{xy}^* + A_{xx}^*D_{xy}^*)\left(\frac{n\pi}{b}\right)^6\right\}\Phi_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$M_{yy} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{(B_{yy}^*B_{yx}^* + A_{yy}^*D_{xy}^*)\left(\frac{m\pi}{a}\right)^6 + [B_{xy}^*B_{yx}^* + B_{yy}^*(B_{xx}^* + B_{yy}^* - 2B_{ss}^*) + D_{xy}^*(2A_{xy}^* + A_{ss}^*) + A_{yy}^*D_{yy}^*]\left(\frac{m\pi}{a}\right)^4\left(\frac{n\pi}{b}\right)^2 + [B_{xy}^*(B_{xx}^* + 2B_{yy}^* - 2B_{ss}^*) + A_{xx}^*D_{xy}^* + D_{yy}^*(2A_{xy}^* + A_{ss}^*)]\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^4 + (B_{xy}^{*2} + A_{xx}^*D_{yy}^*)\left(\frac{n\pi}{b}\right)^6\right\}\Phi_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$M_{xy} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m\pi}{a}\frac{n\pi}{b}\left\{(B_{ss}^*B_{yx}^* - 2A_{yy}^*D_{ss}^*)\left(\frac{m\pi}{a}\right)^4 + [B_{ss}^*(B_{xx}^* + B_{yy}^* - 2B_{ss}^*) - 2D_{ss}^*(2A_{xy}^* + A_{ss}^*)]\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + (B_{ss}^*B_{xy}^* - 2A_{xx}^*D_{xx}^*)\left(\frac{n\pi}{b}\right)^4\right\}\Phi_{mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \quad (15)$$

For a uniformly loaded plate ($p = p_0$ = constant) the Fourier coefficients for the load are

$$p_{mn} = 16p_o/(\pi^2 mn) \quad \text{for } m, n = \text{odd}$$
$$p_{mn} = 0 \quad \text{for all other } m \text{ and } n \quad (16)$$

and the deflection is given by

$$w = \frac{16p_o b^4}{\pi^6(D_{xx}^*D_{yy}^*)^{1/2}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\{w_{mn}\}}{[mn\{\Phi_{mn}\}]}\times\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (17)$$

where

$$\{w_{mn}\} = \alpha^{-1}(m/c)^4 + \gamma n^2(m/c)^2 + \alpha n^4$$
$$\{\Phi_{mn}\} = (m/c)^8(\beta/\alpha + \omega^2) + n^2(m/c)^6(\delta/\alpha + \gamma\beta + 2\chi\omega) + n^4(m/c)^4(\alpha^{-1}\beta^{-1} + \alpha\beta + \gamma\delta + 2\omega\psi + \chi^2) + n^6(m/c)^2(\gamma/\beta + \alpha\delta + 2\psi\chi) + n^8(\alpha/\beta + \psi^2)$$
$$c = a/b$$

Solutions for other loadings can be obtained by expanding the loading in a double Fourier series (11), calculating Φ_{mn} from Eq. (13), and substituting into Eqs. (14) and (15).

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Construction of a Liapunov Function in the Critical Case

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IN this Note, we present a general method for constructing a Liapunov function for a linear system governed by the vector equation

$$dx(t)/dt = Ax(t) \quad (1)$$

where $x(t)$ is a n vector and A is a constant $n \times n$ matrix with distinct and purely imaginary eigenvalues. This system is Liapunov stable,[1] since A is a critical matrix as already

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